JOURNAL OF DIFFERENTIAL EQUATIONS 33, 226–238 (1979)

Regularity Properties of Solutions of an Equation Arising in the Theory of Turbulence

C. J. VAN DUYN

Mathematical Institute, Leiden University, Leiden, Netherlands

Received June 21, 1978

1. INTRODUCTION

In this paper we shall report on some regularity properties of solutions of the Cauchy problem

$$\mathrm{I}\quad \begin{cases} u_t + ([u(0, t) - u(x, t)]^2)_{xx} = 0 & \text{in} \quad S_T = \{(x, t) : x \in \mathbb{R}, t \in (0, T]\}, \quad (1) \\ u(x, 0) = u_0(x) & \text{on} \quad \mathbb{R}, \end{cases}$$

in which t and x denote, respectively a time and a space coordinate, the subscripts t and x denote partial differentiation with respect to these variables and where T may be any positive number. Problem I is the limiting case of the problem:

$$\mathrm{II}\quad \begin{cases} u_t + ([u(0, t) - u(x, t)]^2)_{xx} = \nu u_{xx} & \text{in} \quad S_T, \quad (2) \\ u(x, 0) = u_0(x) & \text{on} \quad \mathbb{R}, \quad (3) \end{cases}$$

which arises in a stochastic model based on the Burgers equation. In (2), $\nu \geqslant 0$; see Frisch, Lesieur and Brissaud [5]. In view of the physical interpretation of this problem it is natural to assume that the function u_0 is positive definite ($u_0 \gg 0$).

Recently, Problems I and II were studied by Brauner, Penel and Temam [2] in the half plane $S = \{(x, t): x \in \mathbb{R}, t > 0\}$. They gave a first rigorous mathematical treatment, proceeding essentially as follows: first they considered Problem II for $\nu > 0$, and they showed, amongst other things, that if $u_0 \in H^1(\mathbb{R})$, there exists a unique solution u_ν in some weak sense. Then, choosing a sequence $\{\nu_k\}$ such that $\nu_k \to 0$ as $k \to \infty$, they showed that $u_{\nu_k} \to u$ as $k \to \infty$ in $L^\infty([0, \infty]; H^1(\mathbb{R}))$ weak star and that u is a solution of Problem I in an appropriate sense. In addition it was shown that for all $\nu \geqslant 0$, $\| u_\nu \|_{L^\infty(S)} \leqslant \| u_0 \|_{L^\infty(\mathbb{R})}$ and that $u_\nu(\cdot, t) \gg 0$ for fixed $t \geqslant 0$.

In a second paper [10] Penel showed that if $u_0 \in C^\infty(\mathbb{R})$, and $\nu > 0$, then $u_\nu \in C^\infty(\bar{S})$.

0022-0396/79/080226-13$02.00/0

Assuming for the moment $u(\cdot, t) \geqslant 0$ for all $t \in [0, T]$, we notice that equation (1) is degenerate parabolic: i.e. at points in S_T where $u(0, t) > u(x, t)$ it is parabolic, but at points where $u(0, t) = u(x, t)$ it is not. Since this type of degeneracy also occurs in the porous media equation ($u_t = (u^m)_{xx}$, $m > 1$) we are led to consider existence of solutions of Problem I in a class of functions, which is analogous to the class of generalized solutions of the porous media equation, defined by Oleinik, Kalashnikov and Yui-Lin [9].

DEFINITION. A function u: $\bar{S}_T \to \mathbb{R}$ is called a generalized solution of Problem I if:

(i) u is continuous and bounded in $\bar{S}_T$;

(ii) $u(\cdot, t) \geqslant 0$ for all $t \in [0, T]$;

(iii) $[u(0, t) - u(x, t)]^2$ possesses a bounded generalized derivative with respect to x in S_T;

(iv) u satisfies the identity:

$$\iint_{S_T} \{\Phi_x(v^2)_x + \Phi_t u\}\, dx\, dt = -\int_{-\infty}^{+\infty} u_0(x)\, \Phi(x, 0)\, dx,$$

for all $\Phi \in C^1(\bar{S}_T)$ which vanish for large $|x|$ and for $t = T$. Here $v(x, t) = u(0, t) - u(x, t)$.

In section 2, we shall prove that if $u_0 \in L^2(\mathbb{R})$, $u_0 \geqslant 0$ and u_0 is uniformly Lipschitz continuous on $\mathbb{R}$, then Problem I has at least one solution u which satisfies (i)–(iv). We shall do this by approximating u_0 by a family of functions $u_{0,n} \in C^\infty(\mathbb{R})$ and considering the corresponding solutions u_n of Problem II for $\nu = 1/n$. This family is then shown to be equicontinuous, and a sequence is shown to converge to u, uniformly on compact subsets of S_T.

In section 3 we shall discuss some regularity properties of the solution u. In particular we show that u is uniformly Lipschitz continuous with respect to x in S_T and that $\partial v^{1+\lambda}/\partial x$ is continuous in S_T for each $\lambda > 0$. By constructing an explicit generalized solution of Problem I, we show that these regularity results are optimal.

Finally in section 4, we show that if u_0 satisfies the inequalities

$$1 - Ax^2 \leqslant u_0(x) \leqslant 1 \qquad \text{for} \quad |x| \leqslant l,$$

where A and l are positive constants, then there exists a constant T_0 such that $u(0, t) = 1$ for $t \in [0, T_0]$. A similar result holds for solutions of the porous media equation [7].

2. Existence of Generalized Solutions

We shall prove the existence of a generalized solution by using a constructive method. First we approximate the initial value u_0 by a family of functions $\{u_{0,n}(x): n \geqslant 1\}$, where all the $u_{0,n}$ have such properties, that if we consider $u_{0,n}$ as an initial value in Problem II, with $\nu = 1/n$, then this problem has a unique classical solution, denoted by $u_n(x, t)$.

There after, we show that the set $\{u_n(x, t): n \geqslant 1\}$ is equicontinuous on $\bar{S}_T$, so that we can use the Arzela–Ascoli Theorem to obtain convergence of some subsequence. Finally we show that the limit function has all the properties required of a generalized solution.

We start with the following lemma.

Lemma 1. *Let $u_0 \in L^2(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R})$, with Lipschitz constant L, and let $u_0 \geqslant 0$. Then there exists a sequence of functions $\{u_{0,n}(x)\}_{n=1}^{\infty}$ in $C^\infty(\mathbb{R})$ such that*

(i) $u_{0,n} \to u_0$ *as* $n \to \infty$, *uniformly on* $\mathbb{R}$;

(ii) $|u_{0,n}'| \leqslant L$ *for all* $n \geqslant 1$;

(iii) $u_{0,n} \in H^1(\mathbb{R})$ *for all* $n \geqslant 1$;

(iv) $u_{0,n} \geqslant 0$ *for all* $n \geqslant 1$;

(v) $|u_{0,n}| \leqslant \sup_{x\in\mathbb{R}} |u_0(x)| = u_0(0)$ *for all* $n \geqslant 1$.

Proof. Let $\rho(x) = e^{-x^2}$ and set, for all $n \geqslant 1$, $\rho_{1/n}(x) = \pi^{-1/2} n\rho(nx)$. Then define the functions $u_{0,n}$, $n \geqslant 1$, by

$$u_{0,n}(x) = \int_{\mathbb{R}} \rho_{1/n}(x - y)\, u_0(y)\, dy,$$

from which the properties (i)–(v) easily follow.

Next we shall consider the functions $u_{0,n}$ as initial values for equation (2): i.e. for a given $n \geqslant 1$, we consider the Cauchy problem

$$\mathrm{II}' \quad \begin{aligned} &u_t = \frac{1}{n} u_{xx} - ([u(0, t) - u(x, t)]^2)_{xx} &&\text{in } S_T, \qquad (4)\\ &u(x, 0) = u_{0,n}(x) &&\text{on } \mathbb{R}. \end{aligned}$$

Then, because $u_{0,n} \in C^\infty(\mathbb{R}) \cap H^1(\mathbb{R})$ and $u_{0,n} \geqslant 0$, it follows from [10] and [2, Theorem 1.2] that Problem II′ has a unique classical solution, denoted by $u = u_n(x, t)$, such that $u_n \in C^\infty(\bar{S}_T)$ and, for each $t \in [0, T]$, $u_n(\cdot, t) \in H^1(\mathbb{R})$ and $u_n(\cdot, t) \geqslant 0$.

In order to prove that the set $\{u_n(x, t): n \geqslant 1\}$ is equicontinuous on $\bar{S}_T$, we shall first derive an estimate on the derivative u_{n_x}, which is uniform with respect to $n \geqslant 1$. We shall make this the content of the next lemma.

LEMMA 2. *Let u_n be the solution of Problem* II′, *in which $u_{0,n}$ satisfies the properties of Lemma* 1. *Then there exists a constant $K \in (0, \infty)$ such that*

$$|(u_n(x, t))_x| \leqslant K,$$

for all $(x, t) \in \bar{S}_T$ and for all $n \geqslant 1$.

Proof. Fix $x_0 \in \mathbb{R}$, and denote by R the rectangle $(x_0 - 1, x_0 + 1) \times (0, T]$ and by R_δ the rectangle $(x_0 - 1 + \delta, x_0 + 1 - \delta) \times (0, T]$, where $\delta \in (0, 1)$.

We observe, using the maximum principle [2], that

$$\sup_R |u_n| \leqslant \sup_{\mathbb{R}} |u_{0,n}| \leqslant u_0(0) \qquad \text{for all} \quad n \geqslant 1.$$

Then, if we write equation (4) as

$$u_t = \{1/n + 2[u(0, t) - u(x, t)]\}u_{xx} - 2u_x^2 \qquad \text{on } R, \tag{5}$$

we can apply a standard Bernstein argument (Aronson, [1]) to equation (5).

Thus, we introduce a new variable $u_n = \phi(w_n)$, where we choose $\phi(r) = 2M(e^r - 1)$ and $M = u_0(0)$, and derive a differential equation for w_{n_x}. Further, let $\xi \in C_0^2([x_0 - 1, x_0 + 1])$ such that $0 \leqslant \xi \leqslant 1$, $\xi = 1$ on $[x_0 - 1 + \delta, x_0 + 1 - \delta]$ and set $z_n(x, t) = \xi^2 w_{n_x}^2$. Then, if z_n attains its maximum value at the lower boundary of R, we have

$$\sup_{R_\delta} z_n \leqslant z_n(\tilde{x}, 0), \qquad \text{where} \quad \tilde{x} \in (x_0 - 1, x_0 + 1).$$

Hence

$$\sup_{R_\delta} \xi \, | w_{n_x} | \leqslant \xi(\tilde{x}) \, | w_{n_x}(\tilde{x}, 0)|.$$

However, since $\xi = 1$ in R_δ and since $u_{n_x} = \phi'(w_n)w_{n_x}$, we find

$$\sup_{R_\delta} | u_{n_x} | \leqslant \frac{\sup \phi'}{\inf \phi'} \, | u'_{0,n} | \leqslant 3L, \tag{6}$$

for all $n \geqslant 1$, and for all $x_0 \in \mathbb{R}$.

If, on the other hand, z attains its maximum value at an interior point $(\tilde{x}, \tilde{t})$ of R, we have $z_{n_x} = 0$, $z_{n_t} \geqslant 0$ and $z_{n_t} - \{1/n + 2[\phi(w_n(0, t)) - \phi(w_n(x, t))]\}z_{n_{xx}} \geqslant 0$ at $(\tilde{x}, \tilde{t})$. Combining these inequalities with the differential equation for w_{n_x}, we obtain an upper bound for z in R, which is independent of n, T and the location of the rectangle R. Moreover since $\xi = 1$ in R_δ, we find

$$\sup_{R_\delta} | w_{n_x} | \leqslant C.$$

Then using

$$\sup_{R_\delta} | u_{n_x} | \leqslant \sup \phi' \sup_{R_\delta} | w_{n_x} | = 3M \sup_{R_\delta} | w_{n_x} |,$$

we obtain

$$\sup_{R_\delta} | u_{n_x} | \leqslant 3MC \tag{7}$$

for all $n \geqslant 1$, and for all $x_0 \in \mathbb{R}$. Finally, combining (6) and (7) we obtain the desired inequality with $K = 3 \max\{L, MC\}$.

Next we write equation (5) as

$$u_t = a(x, t)u_{xx} + b(x, t)u_x \qquad \text{on } S_T, \tag{8}$$

where

$$a(x, t) = \frac{1}{n} + 2[u_n(0, t) - u_n(x, t)] \geqslant \frac{1}{n} > 0,$$

and

$$b(x, t) = -2u_{n_x}(x, t).$$

By Lemma 2 and the maximum principle, it follows that there exists a constant $\mu = \max\{(1 + 4M), 2K\}$ such that for all $(x, t) \in S_T$ and for all $n \geqslant 1$:

$$a(x, t) \leqslant \mu \qquad \text{and} \qquad | b(x, t)| \leqslant \mu.$$

Again from Lemma 2, we have for all (x', t), $(x'', t) \in \bar{S}_T$ and for all $n \geqslant 1$:

$$| u_n(x', t) - u_n(x'', t)| \leqslant K | x' - x'' |. \tag{9}$$

These estimates enable us to apply a theorem of *Gilding* [6], about the Hölder continuity of solutions of parabolic equations, and we obtain for solutions of (8), which satisfy (9):

$$| u_n(x, t') - u_n(x, t'')| \leqslant K_0 | t' - t'' |^{1/2}, \tag{10}$$

for all $n \geqslant 1$ and for all (x, t'), $(x, t'') \in \bar{S}_T$, with $| t' - t'' | < \delta_1$. Here, the constant δ_1 only depends on μ, and the constant K_0 only depends on μ and K.

Thus combining (9) and (10), we find that the set $\{u_n(x, t) : n \geqslant 1\}$ is equicontinuous on $\bar{S}_T$. By the Arzela–Ascoli Theorem, it now follows that there exists a sequence $\{u_{n_k}\}$ and a function $u \in C(\bar{S}_T)$ such that $u_{n_k} \to u$ as $n_k \to \infty$, pointwise on $\bar{S}_T$. This convergence is uniform on any bounded subset of $\bar{S}_T$.

It remains to show that the limit function u has all the properties required of a generalized solution, and we shall do this step by step.

(i) We already obtained that $u \in C(\bar{S}_T)$. Moreover, by the maximum principle, $\sup_{S_T} | u_n | = u_{0,n}(0)$ for all $n \geqslant 1$. Hence $\sup_{S_T} | u | = u_0(0)$.

(ii) We know that $u_n(\cdot, t) \gg 0$ for all $n \geqslant 1$ and for all $t \geqslant 0$. By definition, this means that for each finite set of real numbers $\{x_i\}_{i=1}^N$ and for each finite set of complex numbers $\{\alpha_i\}_{i=1}^N$,

$$\sum_{i,j=1}^{N} u_n(x_i - x_j, t)\, \alpha_i \bar{\alpha}_j \geqslant 0, \tag{11}$$

for all $n \geqslant 1$ and for all $t \geqslant 0$. Then, since $u_{n_k}(x, t) \to u(x, t)$ as $n_k \to \infty$, uniformly on bounded subsets of $\bar{S}_T$, it follows from (11) that $u(\cdot, t) \gg 0$ for all $t \geqslant 0$.

(iii) Let Φ be an admissible testfunction with support $\Phi \subset Q$, where Q is a bounded subset of $\bar{S}_T$, and denote the convergent sequence by $\{u_n\}_{n=1}^{\infty}$. Since $|(v_n^2(x, t))_x| = 2[u_n(0, t) - u_n(x, t)]\, |(u_n(x, t))_x|$ is uniformly bounded with respect to $n \geqslant 1$, for all $(x, t) \in Q$, it follows that $\{(v_n^2)_x\}$ is a bounded sequence in $L^2(Q)$. Hence, there exists a subsequence $\{(v_{n_l}^2)_x\}$ and a bounded function $p \in L^2(Q)$ such that

$$(v_{n_l}^2)_x \rightharpoonup p \qquad \text{in} \quad L^2(Q) \qquad \text{as} \quad n_l \to \infty.$$

Now, let $\zeta \in C_0^1(Q)$. Then

$$((v_{n_l}^2)_x, \zeta) \to (p, \zeta) \qquad \text{as} \quad n_l \to \infty, \tag{12}$$

and

$$((v_{n_l}^2)_x, \zeta) = -(v_{n_l}^2, \zeta_x),$$

where (,) denotes the inner product in $L^2(Q)$. But since $u_{n_l} \to u$, uniformly on bounded subsets, we have

$$(v_{n_l}^2, \zeta_x) \to (v^2, \zeta_x) \qquad \text{as} \quad n_l \to \infty. \tag{13}$$

Hence, combining (12) and (13), we find that p is the generalized derivative of v^2.

(iv) Since u_n is a classical solution of equation (4), it follows that

$$\iint_Q \left\{\Phi_x \left[\frac{1}{n} u_{n_x} - (v_n^2)_x\right] - \Phi_t u_n\right\} dx\, dt = \int_{Q \cap \{t=0\}} \Phi(x, 0)\, u_{0,n}(x)\, dx. \tag{14}$$

By Lemma 2:

$$\iint_Q \Phi_x \frac{1}{n} u_{n_x}\, dx\, dt \to 0 \qquad \text{as} \quad n \to \infty.$$

Hence, if we set $n = n_l$ in (14) and let $n_l \to \infty$, we obtain

$$\iint_Q \{\Phi_x (v^2)_x + \Phi_t u\}\, dx\, dt = -\int_{Q \cap \{t=0\}} \Phi(x, 0)\, u_0(x)\, dx.$$

Finally, since Φ had been chosen arbitrary, we proved the following existence theorem:

THEOREM 1. *Let $u_0 \in L^2(\mathbb{R})$ such that u_0 is uniformly Lipschitz continuous on $\mathbb{R}$ and $u_0 \gg 0$. Then there exists at least one generalized solution u of Problem* I.

3. REGULARITY

In this section we shall prove some regularity properties of the generalized solution of Problem I which we have constructed in the previous section.

THEOREM 2. *Let u be the generalized solution of Problem* I *constructed in section* 2. *Then u has the following properties:*

(i) *For any $t \in [0, T]$,*

$$u(\cdot, t) \in L^2(\mathbb{R}) \qquad \text{and} \qquad \| u(t)\|_{L^2(\mathbb{R})} \leqslant \| u_0 \|_{L^2(\mathbb{R})} ;$$

(ii) *$u(x, t)$ is uniformly Lipschitz continuous with respect to x and uniformly Hölder continuous (exponent $\frac{1}{2}$) with respect to t in $\bar{S}_T$;*

(iii) *u is a classical solution of equation* (1) *in a neighbourhood of any point $(x_0, t_0) \in S_T$ with $x_0 \neq 0$;*

(iv) *given any $\lambda > 0$, $\partial v^{1+\lambda}/\partial x$ exists and is continuous in S_T. Moreover, $\partial v^{1+\lambda}(0, t)/\partial x = 0$ for all $t \in (0, T]$.*

Proof. (i) From [2, Lemma II.1 and remark I.2] we find that for each $t \in [0, T]$ and for all $n \geqslant 1$, $\| u_n(t)\|_{L^2(\mathbb{R})} \leqslant \| u_{0,n} \|_{L^2(\mathbb{R})}$. Moreover, since $\| u_{0,n} \|_{L^2(\mathbb{R})} \leqslant \| u_0 \|_{L^2(\mathbb{R})}$, this implies that $\| u_n(t)\|_{L^2(\mathbb{R})} \leqslant \| u_0 \|_{L^2(\mathbb{R})}$ for all $t \in [0, T]$ and for all $n \geqslant 1$. Now, fix $l > 0$ and let I_l denote the interval $(-l, l)$. Then, because $u_n \to u$ as $n \to \infty$, uniformly on I_l, we find

$$\| u \|_{L^2(I_l)} = \lim_{n \to \infty} \| u_n \|_{L^2(I_l)} \leqslant \| u_0 \|_{L^2(\mathbb{R})} .$$

Hence, letting $l \to \infty$, we obtain the desired result.

(ii) This property follows easily from the inequalities (9) and (10).

(iii) Let (x_0, t_0) be a point in S_T. Then, because $u(\cdot, t_0) \gg 0$,

$$u(x_0, t_0) \leqslant u(0, t_0).$$

We assert that if $x_0 \neq 0$, the inequality is strict. For suppose

$$u(x_0, t_0) = u(0, t_0),$$

for $x_0 \neq 0$. Then, because $u(\cdot, t_0) \gg 0$, $u(x, t_0)$ is periodic with respect to x with period x_0 [2]. However, because $u(\cdot, t_0) \in L^2(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R})$, with Lipschitz constant K, $u(x, t_0) \to 0$ as $|x| \to \infty$, which contradicts the periodicity of $u(x, t_0)$. Thus

$$u(x_0, t_0) < u(0, t_0)$$

and hence, since $u_n(x, t) \to u(x, t)$ uniformly on bounded subsets of S_T, there exists a constant $\epsilon > 0$ and an integer $N(\epsilon)$ such that for all $n > N(\epsilon)$

$$u_n(x_0, t_0) < u_n(0, t_0) - \epsilon.$$

Moreover, since the set $\{u_n(x, t), n \geqslant 1\}$ is equicontinuous, there exists a neighbourhood $N_0 \subset S_T$ of (x_0, t_0) such that

$$u_n(x, t) < u_n(0, t) - \epsilon/2,$$

for all $(x, t) \in N_0$ and for all $n > N(\epsilon)$.

Next observe that $u_{n_x} = q$ satisfies

$$q_t = (a(x, t)q_x)_x + 2b(x, t)q_x \qquad \text{in } N_0, \tag{15}$$

where $a(x, t)$ and $b(x, t)$ are defined below equation (8). Now, because $\epsilon < a(x, t) < 1 + 4M$ and $|b(x, t)| \leqslant 2K$ for all $(x, t) \in N_0$ and for all $n > N(\epsilon)$, we can apply [8, Theorem 10.1] to equation (15) and obtain that there exists a neighbourhood $N_1 \subset N_0$ of (x_0, t_0) such that for all $n > N(\epsilon)$, $u_{n_x} \in C^\alpha(N_1)$ for some $\alpha \in (0, 1)$, where α and $\| u_{n_x} \|_{C^\alpha(N_1)}$ may be estimated independently of n.

Moreover, using property (ii) of this theorem, we also have $u_n \in C^\alpha(N_1)$ for all $n \geqslant 1$, with $\| u_n \|_{C^\alpha(N_1)}$ independently of n. Hence, $a, b \in C^\alpha(N_1)$ and $\| a \|_{C^\alpha(N_1)}$, $\| b \|_{C^\alpha(N_1)}$ can be bounded uniformly with respect to n.

Then, we apply the linear theory [4, p. 61] to equation 8 and find for a neighbourhood $N_2 \subset N_1$ of (x_0, t_0) that $u_n \in C^{2+\alpha}(N_2)$ and $\| u_n \|_{C^{2+\alpha}(N_2)}$ is uniformly bounded with respect to $n > N(\epsilon)$. Hence $u \in C^{2+\alpha}(N_2)$, and it follows directly from the integral identity (iv) that u is a classical solution of (1) in N_2.

(iv) We proceed as in [1]. First we notice, that if $x \neq 0$, property (iv) follows at once from (iii). Therefore, it remains to prove (iv) in a neighbourhood of the line $x = 0$.

Let $\delta > 0$ and denote by Q_δ the set $\{(x, t): |x| < \delta, 0 < t \leqslant T\}$. Then, since $v_n(x, t) = u_n(0, t) - u_n(x, t)$, we have

$$v_n(0, t) = 0 \qquad \text{and} \qquad 0 \leqslant v_n(x, t) \leqslant K\delta,$$

for all $n \geqslant 1$ and for all $(x, t) \in Q_\delta$. Further, given any $\lambda > 0$, each v_n satisfies

$$v_n^{1+\lambda}(x_1, t_1) - v_n^{1+\lambda}(x_2, t_1) = (1 + \lambda) \int_{x_1}^{x_2} v_n^{\lambda}(x, t_1) \frac{\partial}{\partial x} v_n(x, t_1)\, dx,$$

for all $(x_1, t_1), (x_2, t_1) \in S_T$. Hence, using Lemma 2 and the above remarks, it follows that for all $(x_1, t_1), (x_2, t_1) \in Q_\delta$

$$| v_n^{1+\lambda}(x_1, t_1) - v_n^{1+\lambda}(x_2, t_1)| \leqslant (1+\lambda) K^{1+\lambda}\delta^\lambda | x_1 - x_2 |,$$

and, letting $n \to \infty$,

$$| v^{1+\lambda}(x_1, t_1) - v^{1+\lambda}(x_2, t_1)| \leqslant (1+\lambda) K^{1+\lambda}\, \delta^\lambda \,| x_1 - x_2 |. \tag{16}$$

Thus, if we take $x_2 = 0$, we obtain

$$0 \leqslant v^{1+\lambda}(x, t) \leqslant (1+\lambda) K^{1+\lambda}\, \delta^\lambda \,| x |,$$

for all $(x, t) \in Q_\delta$, and it follows that $\partial v^{1+\lambda}(0, t)/\partial x$ exists and equals zero for all $t \in (0, T]$.

Finally, let $(x_1, t_1) \in Q_\delta$ such that $x_1 \neq 0$. Then from (iii) $\partial v^{1+\lambda}(x_1, t_1)/\partial x$ exists, and, from (16)

$$\left| \frac{\partial v^{1+\lambda}}{\partial x}(x_1, t_1) \right| \leqslant (1+\lambda) K^{1+\lambda}\delta^\lambda. \tag{17}$$

Moreover, because $\partial v^{1+\lambda}(0, t)/\partial x = 0$, (17) holds for all $(x_1, t_1) \in Q_\delta$. Hence, using the factor δ^λ in (17), we obtain that $\partial v^{1+\lambda}/\partial x$ is continuous in Q_δ.

In order to show that Theorem 2 provides the best possible global regularity properties in x, we shall construct an explicit generalized solution of Problem I, whose derivative with respect to x is discontinuous across the line $x = 0$, but which satisfies all the properties of Theorem 2.

Consider the half-space problem

$$\text{III} \quad \begin{array}{ll} u_t + (v^2)_{xx} = 0 & x > 0, t > 0, \\ u(0, t) = U(t+1)^\gamma & t \geqslant 0, \\ u(x, 0) = u_0(x) & x \geqslant 0, \end{array} \tag{18}$$

where U is an arbitrary positive number, and γ and u_0 are quantities which will be determined later.

We look for a similarity solution of the form $u(x, t) = U(t+1)^\gamma f(\eta)$, where η denotes the similarity variable $x \cdot (t+1)^{-\beta}$. Here, $\beta \in \mathbb{R}$, will be chosen later. Then, if we substitute this particular solution into the equation and if we choose γ and β such that $-1 - \gamma + 2\beta = 0$, we find that f should satisfy

$$([f(0) - f(\eta)]^2)'' - r\eta f' + sf = 0, \qquad \eta > 0. \tag{19}$$

Here, a prime denotes differentiation with respect to η and $r = \beta/U$ and $s = (2\beta - 1)/U$. At the boundaries we require

$$f(0) = 1 \qquad \text{and} \qquad f(\infty) = 0. \tag{20}$$

Now, set $g(\eta) = 1 - f(\eta)$. Then (19), (20) lead to the two-point boundary value problem

$$\text{III}' \quad \begin{aligned} &(g^2)'' + r\eta g' + s(1-g) = 0, \qquad \eta > 0, \\ &g(0) = 0 \qquad \text{and} \qquad g(\infty) = 1. \end{aligned} \tag{21}$$

In order to obtain solutions of Problem III′ (or III) which satisfy the smoothness properties of Theorem 2, we shall make the following two choices. First, we choose $r + s = 0$, i.e. $\beta = \frac{1}{3}$. Then, (21) can be written as

$$(g^2)'' + (r\eta(g-1))' = 0, \qquad \eta > 0.$$

or

$$(g^2)' + r\eta(g-1) = C_1, \qquad \eta > 0,$$

where C_1 is constant.

Second, we choose $C_1 = 0$, which implies that

$$(g^2)' + r\eta(g-1) = 0, \qquad \eta > 0.$$

Then, it is easily seen that the function g, defined by

$$\{1 - g(\eta)\}\, e^{g(\eta)} = e^{-(r/4)\eta^2}, \qquad r = \frac{1}{3U}, \tag{22}$$

satisfies the equation and the boundary conditions.

Now, recalling that $f(\eta) = 1 - g(\eta)$, it follows from (22) that $f'(0^+) = -(r/2)^{1/2}$ and $0 < f(\eta) \leqslant \exp(-(r/4)\eta^2)$ for all $\eta \geqslant 0$. We shall denote this solution of (19), (20) by $f^+(\eta)$. By symmetry, we can define a function $f^-(\eta)$ on $(-\infty, 0)$, such that f^- satisfies equation (19) for $\eta < 0$ and $f^-(0) = 1$, $f^-(-\infty) = 0$. Finally, define

$$f(\eta) = \begin{cases} f^+(\eta) & \text{when} \quad \eta \geqslant 0, \\ f^-(\eta) & \text{when} \quad \eta \leqslant 0, \end{cases}$$

and

$$\bar{u}(x, t) = U(t+1)^{-1/3} f(x(t+1)^{-1/3}).$$

The function $\bar{u}$ can easily be shown to satisfy the conditions (i), (iii) and (iv) from the definition of a generalized solution. It remains to show that $\bar{u}(\cdot, t) \gg 0$ for all $t \geqslant 0$.

Because $\bar{u}(\cdot, t) \in L^1(\mathbb{R})$ for all $t \geqslant 0$, we can determine its Fourier-transform and, since $\bar{u}(x, t) = \bar{u}(-x, t)$, it is given by

$$\hat{\bar{u}}(s, t) = 2 \int_0^\infty \cos(sx)\, \bar{u}(x, t)\, dx.$$

Now since $f^{+\prime\prime}(\eta) > 0$, a standard calculation shows that $\hat{u}(s, t) \geqslant 0$ for all $s \in \mathbb{R}$ and for all $t \geqslant 0$. Then, using Bochner's Theorem [3], we have $\bar{u}(\cdot, t) \gg 0$ for all $t \geqslant 0$, so $\bar{u}$ is indeed a generalized solution. Moreover, it is easy to see that this solution $\bar{u}$ satisfies all the properties of Theorem 2.

Thus, we may conclude that Theorem 2 is the optimal global regularity result with respect to x for generalized solutions of Problem I.

4. A Geometrical Property

In this section we shall show that for certain initial values a generalized solution of Problem I remains constant at $x = 0$, for a finite time interval $[0, T_0]$.

PROPOSITION. *Let u be the generalized solution of Problem* I *constructed in section* 2, *and let u_0 satisfy* $1 - Ax^2 \leqslant u_0(x) \leqslant 1$ *for* $|x| \leqslant l$, *where A and l are positive constants. Then* $u(0, t) = 1$ *for* $t \in [0, T_0]$, *where* $T_0 = 1/12 \min\{1/A, l^2/2\}$.

Proof. Let $u_n(x, t)$ be the solution of Problem II′ in $\bar{S}_T$, where the initial value $u_{0,n}$ has been constructed as in Lemma 1. Then, if u_0 satisfies the assumption of the proposition, there exist constants $\epsilon(n) > 0$, with $\epsilon(n) \to 0$ as $n \to \infty$, such that

$$1 - \epsilon(n) - Ax^2 \leqslant u_{0,n}(x) \leqslant 1,$$

for all $|x| \leqslant l$ and all $n \geqslant 1$. Now, fix $n \geqslant 1$, and consider Problem II′ in the strip S_{T_0}, where $T_0 > 0$ is a constant which we choose later, and where the initial value is given by the function

$$\tilde{u}_{0,n} = 1 - \epsilon(n) - \frac{x^2}{12T_0}\,.$$

Then, the exact solution is given by

$$u = \tilde{u}_n(x, t) = 1 - \epsilon(n) - \frac{x^2}{12(T_0 - t)} - \frac{1}{6n} \ln \frac{T_0}{T_0 - t}\,.$$

Next we choose for T_0:

$$T_0 = \frac{1}{12} \min \left\{\frac{1}{A}, \frac{l^2}{2}\right\}.$$

Then, if Q_{T_0} denotes the rectangle $(-l, l) \times (0, T_0)$, this choise implies that $\tilde{u}_n \leqslant u_n$ on the parabolic boundary of Q_{T_0}, which we shall denote by Γ_{T_0}.

Set $h(x, t) = u_n(x, t) - \tilde{u}_n(x, t)$. Then h satisfies the equation

$$h_t = ah_{xx} + b_1 h_x + 2\tilde{u}_{n_{xx}}[h(0, t) - h(x, t)] \qquad \text{in} \quad Q_{T_0},$$

and

$$h \geqslant 0 \qquad \text{on} \quad \Gamma_{T_0}.$$

Here, $a = a(x, t)$ is defined below equation (8) and $b_1 = b_1(x, t) = -2[u_{n_x}(x, t) + \tilde{u}_{n_x}(x, t)]$.

Now, let $Q_{t^*} = (-l, l) \times (0, t^*)$, where t^* may be any number in the interval $(0, T_0)$, and let Γ_{t^*} denote its parabolic boundary. Then, for all $t \in (0, t^*)$ we have

$$h(0, t) = u_n(0, t) - \tilde{u}_n(0, t) \leqslant \epsilon(n) + \frac{1}{6n} \ln \frac{T_0}{T_0 - t^*}.$$

Hence, if we set $h(x, t) = y(x, t) + c$, where $c = \epsilon(n) + 1/6n \ln T_0/(T_0 - t^*)$, then y satisfies:

$$y_t \geqslant a y_{xx} + b_1 y_x - 2\tilde{u}_{n_{xx}} y \qquad \text{in} \quad Q_{t^*}, \tag{23}$$

and

$$y \geqslant -\epsilon(n) - \frac{1}{6n} \ln \frac{T_0}{T_0 - t^*} \qquad \text{on} \quad \Gamma_{t^*}.$$

Finally, we can apply a standard maximum principle argument to equation (23) and obtain

$$y(x, t)\,|_{Q_{t^*}} \geqslant -\left\{\epsilon(n) + \frac{1}{6n} \ln \frac{T_0}{T_0 - t^*}\right\} e^{\omega t^*},$$

where ω may be any constant greater then $1/3(T_0 - t^*)$. Hence

$$u_n(x, t) - \tilde{u}_n(x, t) \geqslant -\left\{\epsilon(n) + \frac{1}{6n} \ln \frac{T_0}{T_0 - t^*}\right\} (e^{\omega t^*} - 1)$$

for all $(x, t) \in \bar{Q}_{t^*}$. Thus, letting $n \to \infty$, we find that

$$u(x, t) \geqslant 1 - \frac{x^2}{12(T_0 - t)} \qquad \text{in} \quad \bar{Q}_{t^*},$$

and in particular, $u(0, t) \geqslant 1$ for all $t \in (0, t^*)$. On the other hand, $u(0, t) \leqslant \sup_{x \in \mathbb{R}} u_0(x) = 1$ and hence

$$u(0, t) = 1 \qquad \text{for} \quad t \in [0, t^*].$$

Since $t^* \in (0, T_0)$ was arbitrary, and $u \in C(\bar{S}_T)$,

$$u(0, t) = 1 \qquad \text{for} \quad t \in [0, T_0].$$

Acknowledgment

The author is greatly indebted to Professor L. A. Peletier for his assistence with this work.

References

1. D. G. Aronson, Regularity properties of flows through porous media, *SIAM J. Appl. Math.* **17** (1969), 461–467.
2. C. M. Brauner, P. Penel, and R. Temam, Sur une équation d'évolution non linéaire liée à la théorie de la turbulence, *Ann. Scuola Norm. Sup. Pisa* **4** (1977), 101–128.
3. N. Dunford and J. T. Schwartz, Linear operators—Part II, *in* "Pure and Applied Mathematics VII," Interscience, New York, 1963.
4. A. Friedman, "Partial Differential Equations of Parabolic Type," Prentice-Hall, Englewood Cliffs, N. J., 1964.
5. U. Frisch, M. Lesieur, and A. Brissaud, A Markovian random coupling model for turbulence, *J. Fluid Mech.* **65** (1974), 145–152.
6. B. H. Gilding, Hölder continuity of solutions of parabolic equations, *J. London Math. Soc.* (2), **13** (1976), 103–106.
7. A. S. Kalashnikov, The occurrence of singularities in solutions of the non-steady seepage equation, *USSR Comput. Math. and Math. Phys.* **7** (1967), 269–275.
8. O. A. Ladyzhenskaja, V. A. Solonnikov, and N. N. Ural'ceva, "Linear and Quasilinear Equations of Parabolic Type," Translations of Mathematical Monographs, Vol. 23, Amer. Math. Soc., Providence, R. I., 1968.
9. O. A. Oleinik, A. S. Kalashnikov, and Chzhou Yui-Lin, The Cauchy problem and boundary problems for equations of the type of unsteady filtration, *Izv. Akad. Nauk SSSR Ser. Mat.* **22** (1958), 667–704.
10. P. Penel, Sur une équation d'évolution non linéaire liée à la théorie de la turbulence, II, Existence et régularité de solutions "fortes", to appear.